Analytical approach for solving the radiative transfer equation in two-dimensional layered media

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Abstract

This study presents an analytical approach for obtaining the Green's function of the two-dimensional radiative transfer equation to the boundary-value problem of a layered medium. A conventional Fourier transform and a modified Fourier series which is defined in a rotated reference frame are applied to derive an analytical solution of the radiance in the transformed space. The Monte Carlo method was used for a successful validation of the derived solutions.

Keywords:

1. Introduction

The radiative transfer equation (RTE) is involved in many areas of physics for studying the propagation of waves and particles in scattering media [1, 2] avoiding the high computational cost for solving the Maxwell's equations [3]. The mathematical complexity of this integro-partial differential equation implies that solutions to boundary-value problems are mostly solved numerically by using e.g. the Monte Carlo method [4], the discrete-ordinate method [5], the finite-difference method [6] or the finite-element method [7]. However, it is well-known that analytical approaches have decisive advantages regarding accuracy, speediness and numerical implementation compared to numerical methods and are important for their validation. In the last time a novel analytical approach, the so-called method of rotated reference frame

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(MRRF), was developed [8, 9, 10] for solving the three-dimensional RTE in homogeneous media with planar boundaries such as the slab geometry resulting in a much smaller computational cost than is needed with the conventional spherical harmonics method [1].
In several cases the two-dimensional RTE is used as an appropriate radiative transfer model [5, 11, 12, 13]. Recently, we derived an analytical Green's functions of the two-dimensional RTE for the unbounded medium [14]. The obtained solution has been generalized to homogeneous finite media especially for the circular bounded region and the semi-infinite geometry [15].
In this article an analytical approach for obtaining the Green's function of the RTE in two-dimensional layered media is derived by considering the exact boundary conditions (BC). In particular, the solution for the two-layered medium is discussed in detail. The required computational complexity is, apart from a larger system of linear equations, in the same order of magnitude as needed for the solution of the two-dimensional diffusion equation (DE) for layered media. In addition, the principle proceeding is expandable for solving the three-dimensional RTE in layered media based on the recently derived Green's function reported in [10]. The obtained expressions are successfully verified by comparisons with Monte Carlo simulations.

2. Theory

2.1. General equations

The two-dimensional homogeneous RTE for the radiance $\psi(\boldsymbol{\rho}, \phi)$ in cartesian coordinates is given by

$$\hat{\mathbf{s}} \cdot \nabla \psi(\boldsymbol{\rho}, \phi) + \mu_t \psi(\boldsymbol{\rho}, \phi) = \mu_s \int_0^{2\pi} f(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}') \psi(\boldsymbol{\rho}, \phi') \mathrm{d}\phi', \tag{1}$$

where $\mu_t = \mu_a + \mu_s$ is the total attenuation coefficient, μ_a the absorption coefficient, and μ_s the scattering coefficient. The angle ϕ denotes the direction of the propagation vector $\hat{\mathbf{s}}$ relativ to the x-axes. The phase function $f(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}')$ describes the probability that a particle coming from direction $\hat{\mathbf{s}}'$ is scattered into direction $\hat{\mathbf{s}}$.
In the following it is assumed that an incident beam having an arbitrary direction ϕ_0 enters a N-layered medium, which has an infinite extension regarding the spatial coordinate y, from a non-scattering medium having the same refractive index as all scattering layers, at the boundary $x = 0$. For

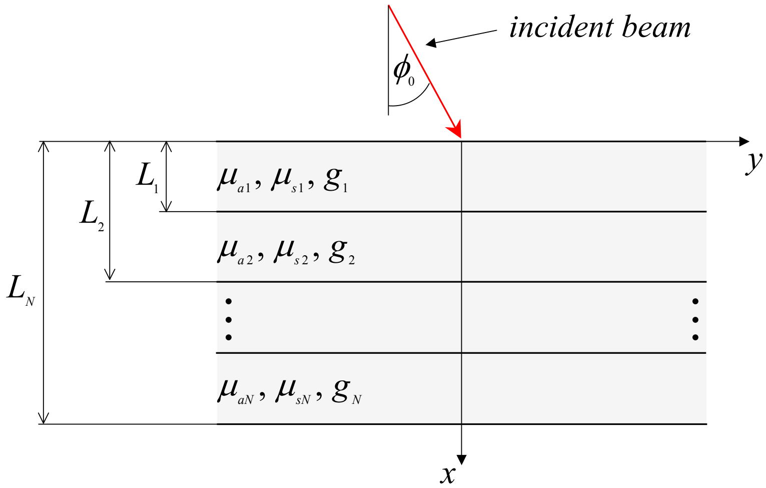

Figure 1: Layered medium.

illustration Fig. 1 shows schematically the geometry of the problem. On the top of the layered medium the radiance must satisfy the inhomogeneous BC

$$\psi(x=0,y,\phi) = S(y)\delta(\phi-\phi_0), \qquad -\pi/2 \le \phi < \pi/2, \tag{2}$$

where $S(y)$ is an arbitrary function for modeling the beam profile. The BC at the plane $x = L_i$ ($i = 1,..,N-1$) between two scattering media requires that the radiance must be continuous for all propagation directions leading to

$$\lim_{x\to L_i^-} \psi(x,y,\phi) = \lim_{x\to L_i^+} \psi(x,y,\phi), \qquad 0 \le \phi < 2\pi. \tag{3}$$

At the bottom $x = L_N$ it is imposed that no radiation enters the scattering medium yielding the homogeneous BC

$$\psi(x=L_N,y,\phi) = 0, \qquad \pi/2 \le \phi < 3\pi/2. \tag{4}$$

2.2. Plane wave solution to the homogeneous RTE

Solutions of differential equations to a given boundary-value problem require knowledge of the general solution to the corresponding homogeneous equations. For obtaining the general solution to the homogeneous Eq. (1) we at first make use of the fact that apart from the condition $\psi(\boldsymbol{\rho},\phi) \to 0$ for $|y| \to \infty$ there are no additional BC regarding the spatial coordinate y.

Therefore, the radiance is expanded in form of the plane-wave decomposition

$$\psi(\boldsymbol{\rho},\phi) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(x,\kappa,\phi)e^{j\kappa y}\mathrm{d}\kappa, \tag{5}$$

where $j = \sqrt{-1}$. Due to this integral transform the homogeneous RTE becomes in the transformed space the form

$$\left(\mu_t + \cos\phi\frac{\partial}{\partial x} + j\kappa\sin\phi\right)\psi(x,\kappa,\phi) = \mu_s\int_0^{2\pi} f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')\psi(x,\kappa,\phi')\mathrm{d}\phi'. \tag{6}$$

For obtaining the solution to a given boundary-value problem we seek a solution in form of the plane-wave mode

$$\psi(x,\kappa,\phi) = e^{\xi x}\psi(\kappa,\phi), \tag{7}$$

where ξ and $\psi(\kappa,\phi)$ are the unknown eigenvalue and eigenfunction, respectively. Substituting this mode in the RTE Eq. (6) yields the eigenvalue problem

$$[\mu_t + k\cos(\phi-\phi_{\mathbf{k}})]\psi(\kappa,\phi) = \mu_s\int_0^{2\pi} f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')\psi(\kappa,\phi')\mathrm{d}\phi', \tag{8}$$

where $k = \sqrt{\mathbf{k}\cdot\mathbf{k}}$ is the norm of the complex wave vector

$$\mathbf{k} = \begin{pmatrix}\xi \\ j\kappa\end{pmatrix} = k\begin{pmatrix}\cos\phi_{\mathbf{k}} \\ \sin\phi_{\mathbf{k}}\end{pmatrix}. \tag{9}$$

For the further derivation all angular dependent quantities of the RTE are expanded in form of the modified Fourier series

$$\psi(\kappa,\phi) = \sum_{m=-\infty}^{\infty}(-1)^m\langle m|u\rangle e^{jm(\phi-\phi_{\mathbf{k}})}, \tag{10}$$

which depends explicitly on the direction $\phi_{\mathbf{k}}$ of the wave vector resulting in a rotated reference frame. The expansion coefficients $\langle m|u\rangle$ are the components of the unknown vector $|u\rangle$. Note that the alternating sign of the expansion coefficients is only due to convenience to receive exactly the same characteristic equation of our previous publication. Thus, the analytical formulae regarding the eigenvector components can be adopted from [14]. The phase function is independent on the direction of the wave vector and, therefore, becomes the conventional series

$$f(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} f_m e^{jm(\phi-\phi')} \tag{11}$$

with the corresponding coefficients

$$f_m = \int_0^{2\pi} f(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}') e^{-jm(\phi-\phi')} \mathrm{d}\phi'. \tag{12}$$

Inserting all series in the eigenvalue problem Eq. (8) results in the characteristic equation

$$k\langle m-1|u\rangle + k\langle m+1|u\rangle - 2\sigma_m\langle m|u\rangle = 0, \tag{13}$$

where $\sigma_m = \mu_a + (1-f_m)\mu_s$. The corresponding solution can be obtained via an eigenvalue decomposition of the following symmetric tridiagonal matrix

$$\mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{T}} = \begin{pmatrix} 0 & \beta_N & 0 & \cdots & \cdots & 0 & 0 \\ \beta_N & \ddots & \ddots & \ddots & \cdots & \cdots & 0 \\ 0 & \ddots & 0 & \beta_1 & 0 & \vdots & \vdots \\ \vdots & \ddots & \beta_1 & 0 & \beta_1 & \ddots & \vdots \\ \vdots & \vdots & 0 & \beta_1 & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & \ddots & \ddots & \ddots & \beta_N \\ 0 & 0 & \cdots & \cdots & 0 & \beta_N & 0 \end{pmatrix}, \tag{14}$$

where $\beta_m = 1/(2\sqrt{\sigma_{m-1}\sigma_m})$. It can be seen that the above tridiagonal matrix only depends on the optical properties of the scattering medium and, therefore, must be decomposed only once. This simplification arises from the introduced modified Fourier series. The use of a conventional Fourier series in a fixed reference frame would lead to a complex and nonsymmetric matrix which has additionally a dependence on the scalar wave number κ resulting in much more computational complexity. Note that for the numerical implementation all Fourier series must be truncated at $|m| \leq N$, where N is an odd number. The matrices $\mathbf{U}$ and $\mathbf{\Lambda}$ contain the orthogonal and normalized eigenvectors $|\nu\rangle$ with components $\langle m|\nu\rangle$ and the real-valued eigenvalues λ, respectively. In the appendix of our previous publication [14] readers can find analytical formulae regarding the eigenvector components of the above tridiagonal matrix for avoiding a numerical iteration of these quantities. Upon determination of the eigenvectors the solution of Eq. (13) is obtained as $\langle m|u\rangle = \langle m|\nu\rangle/\sqrt{\sigma_m}$. The dispersion relation becomes $k = \sqrt{\xi^2 - \kappa^2} = 1/\lambda$ leading to the κ-dependent eigenvalues

$$\xi = \xi(\kappa) = \pm\sqrt{\kappa^2 + \frac{1}{\lambda^2}}. \tag{15}$$

Now, the direction of the wave vector can be obtained from Eq. (9) as

$$\phi_{\mathbf{k}} = \arccos\left(\frac{\xi}{k}\right) = \frac{\pi}{2}[1 - \mathrm{sgn}(\xi)] + j\mathrm{sgn}(\xi)\mathrm{arsinh}(\lambda\kappa). \tag{16}$$

Therefore, the general solution to the homogeneous RTE in the transformed space is given via superposition over all plane-wave modes

$$\psi(x,\kappa,\phi) = \sum_{\lambda_i>0}\left[A_i(\kappa)e^{\xi_i(\kappa)x}\psi_i(\kappa,\phi) + B_i(\kappa)e^{-\xi_i(\kappa)x}\psi_i(-\kappa,\phi+\pi)\right], \tag{17}$$

where the eigenfunctions are obtained as

$$\psi_i(\kappa,\phi) = \sum_{m=-N}^{N}(-1)^m\frac{\langle m|\nu_i\rangle}{\sqrt{\sigma_m}}\exp(jm\phi)\left[\lambda_i\kappa + \sqrt{1+(\lambda_i\kappa)^2}\right]^m. \tag{18}$$

The unknown constants $A_i(\kappa)$ and $B_i(\kappa)$ are obtained from the corresponding BC. The final solution of the boundary-value problem in the transformed space must be integrated according to the inverse Fourier integral Eq. (5).

2.3. Radiative transfer in a two-layered medium

For obtaining the solution of the boundary-value problem in a two-layered medium the radiance in the transformed space is defined as the piecewise-defined function

$$\psi(x,\kappa,\phi) = \begin{cases} \psi_1(x,\kappa,\phi) & 0 \le x \le L_1 \\ \psi_2(x,\kappa,\phi) & L_1 \le x \le L_2 \end{cases}. \tag{19}$$

The radiance within the first layer is governed by the RTE

$$\hat{\mathbf{s}}\cdot\nabla\psi_1(\boldsymbol{\rho},\phi) + \mu_{t1}\psi_1(\boldsymbol{\rho},\phi) = \mu_{s1}\int_0^{2\pi} f_1(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')\psi_1(\boldsymbol{\rho},\phi')\mathrm{d}\phi' \tag{20}$$

and in the second layer by

$$\hat{\mathbf{s}} \cdot \nabla \psi_2(\boldsymbol{\rho}, \phi) + \mu_{t2} \psi_2(\boldsymbol{\rho}, \phi) = \mu_{s2} \int_0^{2\pi} f_2(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}') \psi_2(\boldsymbol{\rho}, \phi') \mathrm{d}\phi'. \tag{21}$$

Based on the last subsection the general solution to the boundary-value problem in the transformed space for the two-layered medium becomes immediately to

$$\psi_n(x, \kappa, \phi) = \sum_{\lambda_{ni} > 0} \left[A_{ni}(\kappa) e^{\xi_{ni}(\kappa) x} \psi_{ni}(\kappa, \phi) + B_{ni}(\kappa) e^{-\xi_{ni}(\kappa) x} \psi_{ni}(-\kappa, \phi + \pi) \right], \tag{22}$$

where

$$\psi_{ni}(\kappa, \phi) = \sum_{m=-N}^{N} (-1)^m \frac{\langle m | \nu_{ni} \rangle}{\sqrt{\sigma_{nm}}} \exp(jm\phi) \left[\lambda_{ni} \kappa + \sqrt{1 + (\lambda_{ni} \kappa)^2} \right]^m \tag{23}$$

and $\sigma_{nm} = \mu_{an} + (1 - f_{mn}) \mu_{sn}$. Note that the index n is only used for convenience to write the solution for both layers simultaneously. The optical properties of the first layer correspond to the index $n = 1$ and that for the second layer to $n = 2$. Therefore, the eigenvalue decomposition of the tridiagonal matrix Eq. (14) must be performed only once for both sets of optical properties. In general, for a N-layered medium the number of matrices which must be decomposed only once is also N. After determination of the eigenvalues and eigenvectors for both sets of optical properties the BC Eq. (2-4) are considered for finding the unknown constants. Applying the Fourier transform leads to the following set of BC

$$\psi(x = 0, \kappa, \phi) = S(\kappa) \delta(\phi - \phi_0), \qquad -\pi/2 \leq \phi < \pi/2, \tag{24}$$

$$\psi(x = L_1^-, \kappa, \phi) = \psi(x = L_1^+, \kappa, \phi), \qquad 0 \leq \phi < 2\pi, \tag{25}$$

$$\psi(x = L_2, \kappa, \phi) = 0, \qquad \pi/2 \leq \phi < 3\pi/2. \tag{26}$$

The whole procedure regarding the determination of the unknown constants are given in the appendix. For the further derivation it is assumed that all required constants of the homogeneous solution are already known.

2.4. Reflectance, transmittance and the fluence

The reflectance, transmittance and the fluence are important quantities for determination of the optical properties of biological tissue. Therefore, this subsection contains the derivation of analytical expressions regarding the above mentioned quantities. First, the fluence in the transformed space within layer n ($n = 1, 2$) is obtained via integration of the radiance as

$$\begin{aligned}\Phi_n(x,\kappa) &= \int_0^{2\pi} \psi_n(x,\kappa,\phi)\mathrm{d}\phi \\ &= \frac{2\pi}{\sqrt{\mu_{an}}} \sum_{\lambda_{ni}>0} \langle 0|\nu_{ni}\rangle \left[A_{ni}(\kappa)e^{\xi_{ni}(\kappa)x} + B_{ni}(\kappa)e^{-\xi_{ni}(\kappa)x}\right]. \end{aligned} \tag{27}$$

The reflectance at the top of the two-layered medium is defined by the half-range integral

$$R(\kappa) = \int_{\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}>0} (\hat{\mathbf{s}}\cdot\hat{\mathbf{n}})\psi(0,\kappa,\phi)\mathrm{d}\phi, \tag{28}$$

where $\hat{\mathbf{n}}$ denotes the outward normal vector. By considering the BC Eq. (24) and $\hat{\mathbf{s}}\cdot\hat{\mathbf{n}} = -\cos\phi$ the above integral can be written as

$$R(\kappa) = \int_{2\pi} (\hat{\mathbf{s}}\cdot\hat{\mathbf{n}})\psi(0,\kappa,\phi)\mathrm{d}\phi - \int_{\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}<0} (\hat{\mathbf{s}}\cdot\hat{\mathbf{n}})S(\kappa)\delta(\phi-\phi_0)\mathrm{d}\phi, \tag{29}$$

yielding the reflectance in the transformed space

$$\begin{aligned} R(\kappa) = S(\kappa)\cos\phi_0 &+ \frac{\pi}{\sqrt{\sigma_{1m}}} \sum_{\lambda_{1i}>0} A_{1i}(\kappa) \\ &\times \left[\langle 1|\nu_{1i}\rangle \left(\lambda_{1i}\kappa + \sqrt{1+(\lambda_{1i}\kappa)^2}\right) + \frac{\langle -1|\nu_{1i}\rangle}{\lambda_{1i}\kappa + \sqrt{1+(\lambda_{1i}\kappa)^2}}\right] \\ &- \frac{\pi}{\sqrt{\sigma_{1m}}} \sum_{\lambda_{1i}>0} B_{1i}(\kappa) \\ &\times \left[\frac{\langle 1|\nu_{1i}\rangle}{\lambda_{1i}\kappa + \sqrt{1+(\lambda_{1i}\kappa)^2}} + \langle -1|\nu_{1i}\rangle \left(\lambda_{1i}\kappa + \sqrt{1+(\lambda_{1i}\kappa)^2}\right)\right]. \end{aligned} \tag{30}$$

Similar as for the reflectance, the transmittance at the bottom can be written by using the BC of Eq. (26) and $\hat{\mathbf{s}} \cdot \hat{\mathbf{n}} = \cos\phi$ as

$$\begin{aligned} T(\kappa) &= \int\limits_{\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}>0} (\hat{\mathbf{s}} \cdot \hat{\mathbf{n}})\psi(L_2, \kappa, \phi)\mathrm{d}\phi = \int\limits_{2\pi} (\hat{\mathbf{s}} \cdot \hat{\mathbf{n}})\psi(L_2, \kappa, \phi)\mathrm{d}\phi \\ &= -\frac{\pi}{\sqrt{\sigma_{2m}}} \sum_{\lambda_{2i}>0} A_{2i}(\kappa) \exp\left[\xi_{2i}(\kappa) L_2\right] \\ &\times \left[\langle 1|\nu_{2i}\rangle \left(\lambda_{2i}\kappa + \sqrt{1 + (\lambda_{2i}\kappa)^2}\right) + \frac{\langle -1|\nu_{2i}\rangle}{\lambda_{2i}\kappa + \sqrt{1 + (\lambda_{2i}\kappa)^2}} \right] \\ &+ \frac{\pi}{\sqrt{\sigma_{2m}}} \sum_{\lambda_{2i}>0} B_{2i}(\kappa) \exp\left[-\xi_{2i}(\kappa) L_2\right] \\ &\times \left[\frac{\langle 1|\nu_{2i}\rangle}{\lambda_{2i}\kappa + \sqrt{1 + (\lambda_{2i}\kappa)^2}} + \langle -1|\nu_{2i}\rangle \left(\lambda_{2i}\kappa + \sqrt{1 + (\lambda_{2i}\kappa)^2}\right) \right]. \end{aligned} \quad (31)$$

3. Numerical results

In this section the obtained solutions are validated against the Monte Carlo method, which converges in the limit of an infinitely large number of simulated photons to the exact solution of the RTE. For the following comparisons the Henyey-Greenstein phase function for two-dimensional media [16]

$$f(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}') = \frac{1}{2\pi} \frac{1 - g^2}{1 + g^2 - 2g\cos(\phi - \phi')} = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} g^{|m|} e^{jm(\phi - \phi')}, \quad (32)$$

is considered. In all cases the asymmetry parameter is assumed as $g = 0.9$ which corresponds to highly forward scattering media such as biological tissue.

For the first comparison the reflectance and the transmittance of a homogeneous slab geometry caused by a perpendicular incident Gaussian beam

$$S(y) = \frac{1}{\sqrt{2\pi}\varepsilon} \exp\left(-\frac{y^2}{2\varepsilon^2}\right), \quad (33)$$

which corresponds to $S(\kappa) = \exp(-\varepsilon^2\kappa^2/2)$, is calculated with the analytical method and the Monte Carlo simulation, see Fig. 2.

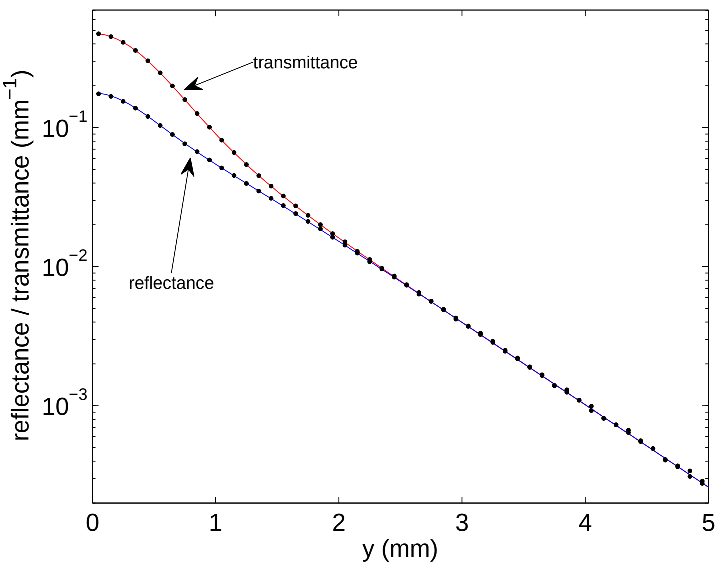

Figure 2: Spatially-resolved reflectance and transmittance for a homogeneous slab caused by a perpendicular incident Gaussian beam with $\varepsilon = 0.3\,\mathrm{mm}$. The optical and geometrical properties are assumed to be $\mu_a = 0.01\,\mathrm{mm}^{-1}$, $\mu_s = 10\,\mathrm{mm}^{-1}$ and $L = 1\,\mathrm{mm}$.

It can be seen that the derived analytical approach (solid curves) and the Monte Carlo method (filled dots) are the same both for the reflectance and the transmittance. Note that for large distances the values of the reflectance converge to those of the transmittance whereas for small distances the reflectance is smaller.

For the next comparison it is assumed that a incident perpendicular to the boundary δ-beam given by $S(y) = \delta(y)$ enters a two-layered medium at the boundary $x = 0$. Fig. 3 shows the resulting transmittance at the bottom $x = L_2$ for two different scattering coefficients of the second layer.

As before the analytical solution (solid curves) agrees well with the transmittance obtained from the Monte Carlo method (noisy curves) for both scattering coefficients.

For the third comparison a two-layered medium having a relatively thin first layer and an infinitely extended second layer is considered. Fig. 4 shows the spatially-resolved reflectance at the top caused by a perpendicular Gaussian beam with parameter $\varepsilon = 0.3\,\mathrm{mm}$ for three different absorption coefficients of the first layer.

The influence of the increasing absorption is well described by the ana-

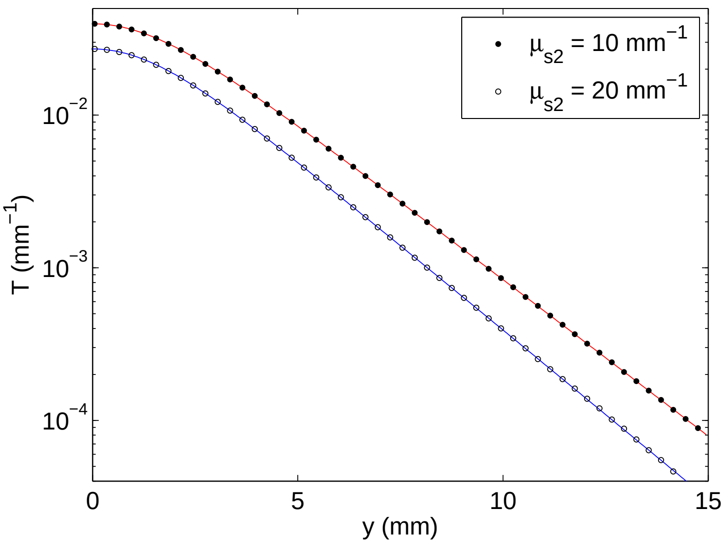

Figure 3: Spatially-resolved transmittance for a two-layered medium caused by a perpendicular incident δ-beam. The optical and geometrical properties are given by $\mu_{a1} = 0.005\,\text{mm}^{-1}$, $\mu_{a2} = 0.01\,\text{mm}^{-1}$, $\mu_{s1} = 6\,\text{mm}^{-1}$, $L_1 = 2\,\text{mm}$ and $L_2 = 3\,\text{mm}$.

lytical approach and the numerical solution.

In Fig. 5 the spatially resolved reflectance caused by a Gaussian beam for different incident directions ϕ_0 and the parameter $\varepsilon = 0.4\,\text{mm}$ is shown for a semi-infinite two-layered medium.

Again, the reflection at the top obtained from the analytical approach and from the Monte Carlo simulation agrees well for all incident directions.

4. Conclusions

In this study the RTE is solved for two-dimensional layered media considering the exact boundary conditions at the interfaces. Especially, the solution for a two-layered medium was explicitly derived and validated against Monte Carlo simulations. Due to the introduced rotated reference frame the obtained radiance in the transformed space is given in terms of analytical functions. The presented method can be applied to obtain the solutions of two-dimensional radiative transfer problems, which are illuminated by an arbitrary external source. For example, an interesting application of the derived equations is the calculation of the spatially resolved reflectance from and fluence in carbon fiber reinforced materials. Further, it can be used to validate

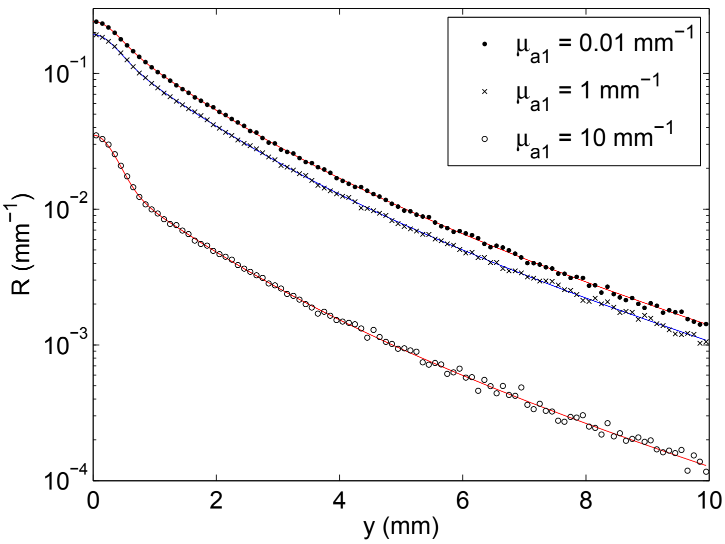

Figure 4: Spatially-resolved reflectance for a semi-infinite two-layered medium with optical and geometrical properties $\mu_{a2} = 0.02\,\mathrm{mm}^{-1}$, $\mu_{s1} = 12\,\mathrm{mm}^{-1}$, $\mu_{s2} = 10\,\mathrm{mm}^{-1}$, $L_1 = 0.1\,\mathrm{mm}$ and $L_2 = \infty$.

the results obtained by numerical methods. Another important aspect is that it can be expanded to the three-dimensional case based on the corresponding Green's function recently derived in [10]. Within the derivations no approximations are made yielding in an excellent agreement compared to the Monte Carlo method. The relative differences between the analytical Green's functions and the Monte Carlo simulations are only due to the stochastic nature of the Monte Carlo method.

5. Appendix: Determination of the unknown constants

In this appendix all required systems of linear equations for finding the unknown constants to the homogeneous solution of the two-layered medium are derived. First, we make use of the inhomogeneous BC, Eq. (2), at the top of the scattering medium. By substitution the homogeneous solution for $n = 1$ in Eq. (24), multiplying both sides with $\exp(-jm'\phi)$ and integrating over the interval $-\pi/2 \leq \phi \leq \pi/2$ yields the first set of equations

$$\sum_{\lambda_{1i}>0} a_{1i,m'}(\kappa) A_{1i}(\kappa) + \sum_{\lambda_{1i}>0} b_{1i,m'}(\kappa) B_{1i}(\kappa) = \frac{1}{\pi} \exp(-jm'\phi_0) S(\kappa). \quad (34)$$

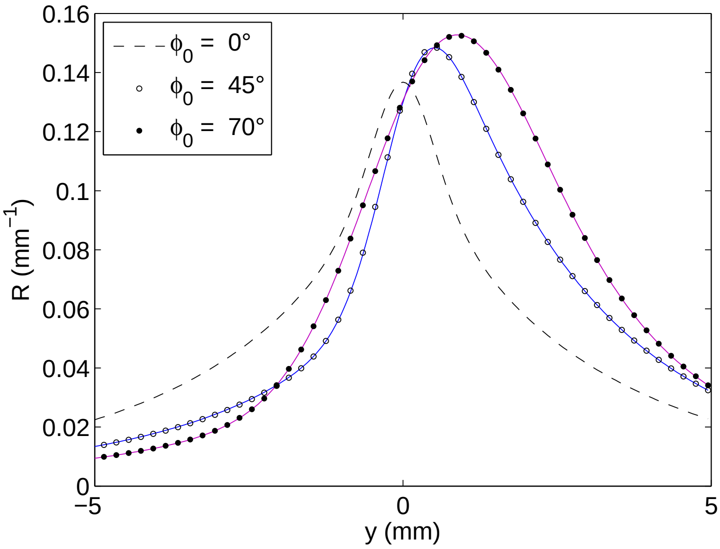

Figure 5: Spatially-resolved reflectance for a semi-infinite two-layered medium with optical and geometrical properties $\mu_{a1} = 0.008\,\text{mm}^{-1}$, $\mu_{a2} = 0.01\,\text{mm}^{-1}$, $\mu_{s1} = 5\,\text{mm}^{-1}$, $\mu_{s2} = 12\,\text{mm}^{-1}$, $L_1 = 5\,\text{mm}$ and $L_2 = \infty$.

Note that all required quantities within the equations and further comments are listed below. The BC between both scattering media Eq. (3) can be splitted in two sections and therefore contains two sets of linear equations. Inserting the homogeneous solution in Eq. (25), multiplying both sides with $\exp(-jm'\phi)$ and integrating over the half-range $-\pi/2 \leq \phi \leq \pi/2$ yields

$$\sum_{\lambda_{1i}>0} a_{1i,m'}(\kappa)\exp[\xi_{1i}(\kappa)L_1]A_{1i}(\kappa) + \sum_{\lambda_{1i}>0} b_{1i,m'}(\kappa)\exp[-\xi_{1i}(\kappa)L_1]B_{1i}(\kappa)$$
$$= \sum_{\lambda_{2i}>0} a_{2i,m'}(\kappa)\exp[\xi_{2i}(\kappa)L_1]A_{2i}(\kappa) + \sum_{\lambda_{2i}>0} b_{2i,m'}(\kappa)\exp[-\xi_{2i}(\kappa)L_1]B_{2i}(\kappa), \tag{35}$$

whereas the integration over $\pi/2 \leq \phi \leq 3\pi/2$ leads to

$$\sum_{\lambda_{1i}>0} c_{1i,m'}(\kappa)\exp[\xi_{1i}(\kappa)L_1]A_{1i}(\kappa) + \sum_{\lambda_{1i}>0} d_{1i,m'}(\kappa)\exp[-\xi_{1i}(\kappa)L_1]B_{1i}(\kappa)$$
$$= \sum_{\lambda_{2i}>0} c_{2i,m'}(\kappa)\exp[\xi_{2i}(\kappa)L_1]A_{2i}(\kappa) + \sum_{\lambda_{2i}>0} d_{2i,m'}(\kappa)\exp[-\xi_{2i}(\kappa)L_1]B_{2i}(\kappa). \tag{36}$$

Finally, the substitution of the homogeneous solution in Eq. (26) at the bottom, multiplying both sides with $\exp(-jm'\phi)$ and the integration over the half-range $\pi/2 \leq \phi \leq 3\pi/2$ gives the fourth set of linear equations

$$\sum_{\lambda_{2i}>0} c_{2i,m'}(\kappa)\exp[\xi_{2i}(\kappa)L_2]A_{2i}(\kappa) + \sum_{\lambda_{2i}>0} d_{2i,m'}(\kappa)\exp[-\xi_{2i}(\kappa)L_2]B_{2i}(\kappa) = 0. \tag{37}$$

The required quantities within the above linear equations are given by

$$a_{ni,m'}(\kappa) = \sum_{m=-N}^{N} \frac{\langle m|\nu_{ni}\rangle}{\sqrt{\sigma_{nm}}} \operatorname{sinc}\left(\frac{m-m'}{2}\right)\left[-\lambda_{ni}\kappa - \sqrt{1+(\lambda_{ni}\kappa)^2}\right]^m \tag{38}$$

$$b_{ni,m'}(\kappa) = \sum_{m=-N}^{N} \frac{\langle m|\nu_{ni}\rangle}{\sqrt{\sigma_{nm}}} \operatorname{sinc}\left(\frac{m-m'}{2}\right)\left[-\lambda_{ni}\kappa + \sqrt{1+(\lambda_{ni}\kappa)^2}\right]^m \tag{39}$$

$$c_{ni,m'}(\kappa) = \sum_{m=-N}^{N} \frac{\langle m|\nu_{ni}\rangle}{\sqrt{\sigma_{nm}}} \operatorname{sinc}\left(\frac{m-m'}{2}\right)\left[\lambda_{ni}\kappa + \sqrt{1+(\lambda_{ni}\kappa)^2}\right]^m \tag{40}$$

$$d_{ni,m'}(\kappa) = \sum_{m=-N}^{N} \frac{\langle m|\nu_{ni}\rangle}{\sqrt{\sigma_{nm}}} \operatorname{sinc}\left(\frac{m-m'}{2}\right)\left[\lambda_{ni}\kappa - \sqrt{1+(\lambda_{ni}\kappa)^2}\right]^m, \tag{41}$$

where $\operatorname{sinc}(x) = \sin(\pi x)/(\pi x)$. The complete system of linear equations is solved by taking equations for even numbers $|m'| = 0, 2, ..., N-1$ resulting in a well-conditioned system of $4N$ linearly independent equations for $4N$ unknown constants. Note that for $L_2 \to \infty$, which leads to a two-layered semi-infinite medium, the BC Eq. (26) on the bottom simplifies to $A_{2i}(\kappa) = 0$.

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