Spatially modulated light source obliquely incident on a semi-infinite scattering medium

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The three-dimensional radiative transfer equation is solved in the spatial frequency domain for modeling the light propagation due to a spatially modulated light source obliquely incident on a semi-infinite uniform medium. The dependence of the derived solution on the spatial frequencies as well as on position and direction is found analytically. The main computational procedure arises from the determination of several constants obtained by a system of linear equations. The obtained equations are verified and illustrated by comparisons with Monte Carlo simulations and the diffusion approximation, respectively.

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The spatial frequency domain (SFD) method is a powerful imaging technique which is based on the projection of structured light on turbid media for noninvasive characterization of the optical properties over a large field of view [1–3]. In recent years a variety of experimental setups as well as theoretical models were developed for performing SFD imaging [4–7]. From a theoretical point of view the propagation of light in random media such as biological tissue is mainly described by the radiative transfer equation (RTE) [8,9]. Up to now solutions to the RTE in the SFD for relevant geometries are obtained via discrete Fourier transformation (DFT) of spatially resolved Monte Carlo (MC) results [6]. Recently, Gardner and Venugopalan introduced an MC method for solving the RTE in the SFD avoiding the use of discrete transforms [7]. However, the currently available analytical solutions and approaches used within SFD imaging are restricted to the diffusion approximation (DA) [5,6,10].

In this Letter we present an analytical radiative transfer model for studying the propagation of spatially modulated light obliquely projected on a semi-infinite medium. The final results obtained in the SFD depend analytically on the spatial frequencies enabling accurate and rapid processing of SFD images. The main computational procedure regarding their evaluation is the solution of several linear equations arising from the considered boundary condition (BC). The derived equations are verified and illustrated by comparisons of the SFD reflectance and the spatial phase shift with MC simulations and the DA, respectively.

The three-dimensional homogeneous RTE for the specific intensity $I(\mathbf{r},\hat{\mathbf{s}})$ in cartesian coordinates is given by [11]

$$\hat{\mathbf{s}}\cdot\nabla I(\mathbf{r},\hat{\mathbf{s}})+\mu_t I(\mathbf{r},\hat{\mathbf{s}})=\mu_s\int f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')I(\mathbf{r},\hat{\mathbf{s}}')\mathrm{d}^2s', \qquad (1)$$

where $\mu_t=\mu_a+\mu_s$ is the total attenuation coefficient, μ_a the absorption coefficient and μ_s the scattering coefficient. The unit vector $\hat{\mathbf{s}}$ specifies the direction of the wave propagation and the phase function $f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')$ describes the probability that a particle coming from the direction $\hat{\mathbf{s}}'$ is scattered into the direction $\hat{\mathbf{s}}$.

Here it is supposed that the semi-infinite medium is bounded by the plane $z=0$ with the corresponding outward normal vector $\hat{\mathbf{n}}=-\hat{\mathbf{z}}$. For modeling the light propagation in the scattering half-space $z\geq 0$ Eq. (1) is solved subject to the inhomogeneous BC

$$I(\boldsymbol{\rho},z=0,\hat{\mathbf{s}})=\exp(i\mathbf{k}\cdot\boldsymbol{\rho})\delta(\hat{\mathbf{s}}-\hat{\mathbf{s}}_0), \qquad \hat{\mathbf{s}}\cdot\hat{\mathbf{z}}>0, \qquad (2)$$

where $\mathbf{k}=(k_x,k_y)$ is the wave vector of spatial modulation and $\hat{\mathbf{s}}_0$ describes the direction of the incident light source $I_{\mathrm{inc}}(\boldsymbol{\rho},\hat{\mathbf{s}})$. The caused specific intensity exhibits the same spatial modulation and takes the form

$$I(\mathbf{r},\hat{\mathbf{s}})=\exp(i\mathbf{k}\cdot\boldsymbol{\rho})I(z,\hat{\mathbf{s}}), \qquad (3)$$

where $I(z,\hat{\mathbf{s}})$ is the corresponding Fourier component. Inserting (3) in (1) and (2) yields the RTE in the SFD

$$\cos\theta\frac{\partial I(z,\hat{\mathbf{s}})}{\partial z}=-[ik\sin\theta\cos(\phi-\phi_{\mathbf{k}})+\mu_t]I(z,\hat{\mathbf{s}}) + \mu_s\int f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')I(z,\hat{\mathbf{s}}')\mathrm{d}^2s' \qquad (4)$$

with the associated BC

$$I(z=0,\hat{\mathbf{s}})=\delta(\hat{\mathbf{s}}-\hat{\mathbf{s}}_0), \qquad \hat{\mathbf{s}}\cdot\hat{\mathbf{z}}>0. \qquad (5)$$

Next, seeking solutions in form of the plane-wave mode

$$I(z,\hat{\mathbf{s}})=e^{\xi z}I(\hat{\mathbf{s}}), \qquad (6)$$

where ξ and $I(\hat{\mathbf{s}})$ are the unknown eigenvalue and eigenfunction, respectively, one obtains the following eigenvalue problem

$$[\hat{\mathbf{s}}\cdot\mathbf{Q}+\mu_t]I(\hat{\mathbf{s}})=\mu_s\int f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')I(\hat{\mathbf{s}}')\mathrm{d}^2s' \qquad (7)$$

with the three-dimensional vector $\mathbf{Q}=(ik_1,ik_2,\xi)$. In order to solve this eigenvalue problem we take advantage

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of our recent study [12] and hence arrive at the general solution for the Fourier component of the specific intensity

$$I(z,\hat{\mathbf{s}}) = \sum_{\lambda_i>0} C_i(\mathbf{k}) \exp[-\mathcal{Q}_i(k)z] \times \sum_{l=M}^{N} \sum_{m=-l}^{l} \psi_{lm}(k\lambda_i) Y_{lm}(\hat{\mathbf{s}}) \exp(-im\phi_{\mathbf{k}}), \quad (8)$$

where $\mathcal{Q}_i(k) = \sqrt{k^2 + 1/\lambda_i^2}$ and $Y_{lm}(\hat{\mathbf{s}})$ are the spherical harmonics. Here we note that all remaining quantities such as λ_i and $\psi_{lm}(k\lambda_i)$ as well as a detailed derivation of the above expression are explicitly given in [12]. The unknown constants $C_i(\mathbf{k})$ are to be determined from the BC (5). This leads to the following system of linear equations labeled by the values $m' = 0, 1, \ldots, N-1$ and $l' = m'+1, m'+3, \ldots, N$

$$\sum_{\lambda_i>0} C_i(\mathbf{k}) \sum_{l=\bar{l}}^{N} \psi_{lm'}(k\lambda_i) R_{l'l}^{m'} = \mathrm{Re}\{Y_{l'm'}(\theta_0, \phi_0 - \phi_{\mathbf{k}})\}, \quad (9)$$

which was also explicitly derived in [12]. It can be seen that the direction $\hat{\mathbf{s}}_0$ of the incident light source appears only in the right-hand side of the matrix equation (9). Thus, the computational cost for obliquely incident sources is practically the same as in the case of perpendicular illumination. Finally, the SFD reflectance $R(\mathbf{k})$ and the internal density $\Phi(\mathbf{k}, z)$ as function of the spatial frequencies $\mathbf{k}$ are obtained via integration of the specific intensity [12] leading to

$$R(\mathbf{k}) = \cos\theta_0 - \sqrt{\frac{4\pi}{3}} \sum_{i=1}^{N} C_i(\mathbf{k}) \psi_{10}(k\lambda_i), \quad (10)$$

$$\Phi(\mathbf{k}, z) = \sqrt{4\pi} \sum_{i=1}^{\frac{N+1}{2}} C_i(\mathbf{k}) \psi_{00}(k\lambda_i) \exp[-\mathcal{Q}_i(k)z]. \quad (11)$$

The above derived expressions are also illustrated by comparisons to the DA. The SFD reflectance in DA caused by an obliquely incident modulated source with $k_y = 0$ and $\phi_0 = 0$ was developed in [5] and is given by

$$R(k_x) = \frac{\mu_s'}{1+2QD} \frac{\mu_t' + Q\cos\theta_0 + ik\sin\theta_0}{(\mu_t' + Q\cos\theta_0)^2 + (k\sin\theta_0)^2}, \quad (12)$$

where $\mu_t' = \mu_a + \mu_s'$, $D = 1/(3\mu_t')$ and $Q = \sqrt{k_x^2 + \mu_a/D}$. Note that the above expression is based on the solution of the SFD diffusion equation

$$D\frac{d^2\Phi(z)}{dz^2} - (\mu_a + Dk^2)\Phi(z) = -S(z) \quad (13)$$

subject to the partial current BC [9] and an exponentially decreasing source term.

The derived solution to the RTE is verified by comparisons with MC simulations [13]. An analog Monte Carlo method was used applying typically 5 million photons for each simulation. In the following the Henyey–Greenstein phase function [9]

$$f(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}') = \frac{1}{4\pi} \frac{1-g^2}{[1+g^2 - 2g(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}')]^{3/2}} \quad (14)$$

with anisotropy parameter $g = 0.9$ is considered. The incident light source is spatially modulated along the x direction with k_x whereas the illumination along the y direction is assumed to be constant which entails $k_y = 0$. Thus, the resulting incident light source at the interface becomes $I_{\mathrm{inc}}(\boldsymbol{\rho}, \hat{\mathbf{s}}) = \exp(ikx)\delta(\hat{\mathbf{s}} - \hat{\mathbf{s}}_0)$ with $k = k_x$. The MC results in the SFD are obtained via a DFT of the spatially resolved reflectance generated by an incident δ-function source. The spatial resolution in the MC simulation was $\Delta x = 0.025$ mm. In all figures below the derived analytical approach is denoted by solid curves whereas the MC results and the DA are depicted with symbols and dashed lines, respectively.

Figure 1 shows the SFD reflectance versus spatial frequency caused by a spatially modulated light source normally incident on the semi-infinite medium for three different values of the scattering coefficient. It can be seen that the reflectance obtained by the analytical solution of the RTE agrees with the MC results, whereas the DA delivers significant differences. These differences are mainly due to the photons which are reflected close to the incident beam and which cannot be properly described by the DA.

Next, the SFD reflectance caused by a perpendicular illumination of the half-space medium is computed for two relatively large absorption coefficients and shown in Fig. 2. Analogously to Fig. 1, the MC simulations agree with the analytical RTE solutions. However, due to the relatively large absorption the relative differences of the DA are increased.

Next, we consider a spatially modulated light source obliquely incident on the semi-infinite medium. The polar and azimuthal angles of incidence are assumed to be $\theta_0 = \pi/3$ and $\phi_0 = 0$, respectively. The amplitude $|R(\mathbf{k})|$ of the SFD reflectance and the spatial phase shift are shown in Fig. 3 for the absorption coefficient $\mu_a = 0.005\ \mathrm{mm}^{-1}$ and two different scattering coefficients. It can be seen in Fig. 3 that both solutions of the RTE agree

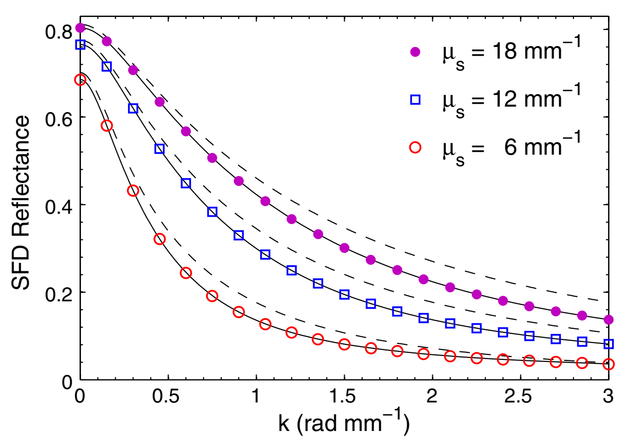

Fig. 1. (Color online) Reflectance versus spatial frequency due to a modulated light source that is incident perpendicularly on a semi-infinite medium having an absorption coefficient of $\mu_a = 0.01\ \mathrm{mm}^{-1}$.

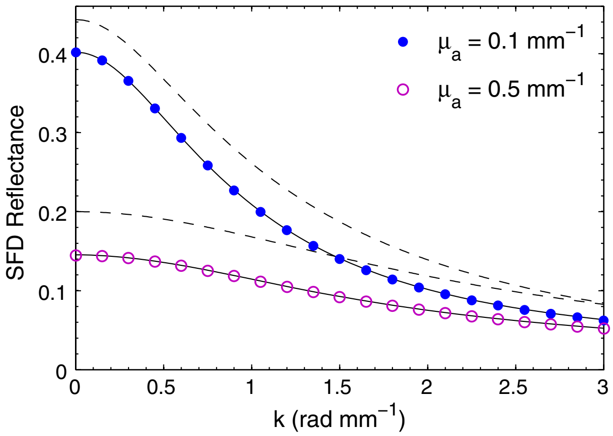

Fig. 2. (Color online) Reflectance versus spatial frequency due to a modulated light source that is incident perpendicularly on a semi-infinite medium having a scattering coefficient of $\mu_s = 10\ \mathrm{mm}^{-1}$.

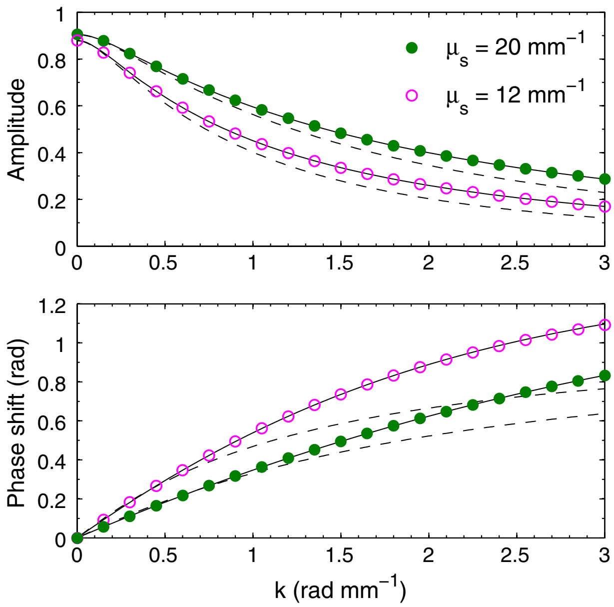

Fig. 3. (Color online) Reflectance amplitude and spatial phase shift versus spatial frequency caused by a modulated light source obliquely incident on the semi-infinite medium.

well, but the DA shows significant differences especially for the spatial phase shift.

In conclusion, we have derived an exact solution of the RTE in the SFD which is more precise compared to the currently available analytical methods used within SFD imaging. To this end the RTE in the SFD was solved for the boundary-value problem in the semi-infinite geometry where we also considered results of our recent publication [12]. The obtained expressions for the specific intensity, the SFD reflectance and the internal density are given in terms of analytical functions such as the Wigner d function enabling a rapid and accurate evaluation, which is especially important for solving inverse problems. The calculation time for one spatial frequency $\mathbf{k} = (k, \phi_{\mathbf{k}})$ is in the order of several milliseconds using a MATLAB script. Numerical results for the SFD reflectance were verified with MC simulations. In addition, the results were compared to the DA.

We also note that the derived method is not restricted to a semi-infinite geometry. It can be extended for example to a layered radiative transport model considering, additionally, the BCs between the interfaces of the optical different layers. In addition, the derived solutions for the semi-infinite geometry can be applied to a pertubation approach in order to consider z-dependent scattering or absorbing inhomogeneities [9]. Further, the obtained solutions can also be used for the calculation of the propagation of fluorescent light.

References

1. D. J. Cuccia, F. P. Bevilacqua, A. J. Durkin, and B. J. Tromberg, Opt. Lett. **30**, 1354 (2005).
2. A. Bassi, C. D'Andrea, G. Valentini, R. Cubeddu, and S. Arridge, Opt. Lett. **33**, 2836 (2008).
3. V. Lukic, V. A. Markel, and J. C. Schotland, Opt. Lett. **34**, 983 (2009).
4. C. D'Andrea, N. Ducros, A. Bassi, S. Arridge, and G. Valentini, Biomed. Opt. Express **1**, 471 (2010).
5. A. Bassi, D. J. Cuccia, A. J. Durkin, and B. J. Tromberg, J. Opt. Soc. Am. A **25**, 2833 (2008).
6. D. J. Cuccia, F. P. Bevilacqua, A. J. Durkin, F. R. Ayers, and B. J. Tromberg, J. Biomed. Opt. **14**, 024012 (2009).
7. A. R. Gardner and V. Venugopalan, Opt. Lett. **36**, 2269 (2011).
8. A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic Press, New York, 1978).
9. F. Martelli, S. Del Bianco, A. Ismaelli, and G. Zaccanti, *Light Propagation through Biological Tissue* (SPIE Press, Bellingham, 2010).
10. J. R. Weber, D. J. Cuccia, A. J. Durkin, and B. J. Tromberg, J. Appl. Phys. **105**, 102028 (2009).
11. K. M. Case and P. F. Zweifel, *Linear Transport Theory* (Addison-Wesley, New York, 1967).
12. A. Liemert and A. Kienle, J. Opt. Soc. Am. A **29**, 1475 (2012).
13. A. Kienle, "Lichtausbreitung in biologischem Gewebe," dissertation (University of Ulm, 1995).