Novel analytical solution for the radiance in an anisotropically scattering medium

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We report on a novel analytical solution of the three-dimensional radiative transport equation for the case of an infinitely extended anisotropically scattering medium that is illuminated by an isotropic point light source. The resulting expression for the radiance can be evaluated efficiently and accurately and exhibits significant improvements with respect to the convergence and the numerical stability compared to the solutions found in the literature so far. The equations obtained were successfully verified by comparisons with the Monte Carlo method.

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1. Introduction

The radiative transport equation (RTE) and its approximations are important for many areas of physical sciences such as tissue optics [1–5], ocean optics [6,7], astrophysics [8], heat transfer [9], and neutron transport theory [10–12]. For example, interstitial radiance measurements were performed recently for characterization of biological tissue by measuring the light intensity for multiple detection angles at a single spatial distance to an isotropic light source [13,14]. The advantage of the radiance spectroscopy compared to fluence measurements can be reviewed in [13]. The underlying theoretical model regarding the interstitial radiance measurements was an isotropic point source embedded in an infinite scattering medium. This model was also considered by Faris [15] for determination of the optical properties of turbid media from frequency-domain measurements [16].

The derivation of accurate, efficient, and numerically stable solutions to the RTE is, even for the simplest geometries, a challenging task. For example, an accurate and reliable solution for the radiance near an isotropic point source in an infinitely extended anisotropically scattering medium seems to be an open problem that is usually solved via time-consuming Monte Carlo (MC) simulations. In this article we consider the isotropic point source problem and report on an analytical solution that exhibits significant improvements with respect to the accuracy and numerical stability over the existing solutions found in the literature. The reason for the resulting improvements related to the convergence and numerical stability is, e.g., due to the treatment of the source term as well as the performance of a modified Fourier inversion. In addition to the problem treated in this paper, novel analytical solutions of the RTE for infinite media have been recently reported and published in [17–19]. For example, in [18] the derivation of single-scattering solutions to the RTE in infinite turbid media is given, whereas Machida presents an extension of Case's method up to three dimensions [19]. It should also be mentioned that the diffusion approximation (DA) shows, in the case of the radiance, significant differences from the exact transport theory solution even for

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large source–detector separations, so it cannot be successfully applied as an approximation of the more complicated RTE. Recently derived RTE solutions [20,21] dealing with the isotropic point source lead to significant improvements over the standard DA provided that the distance to the isotropic point source is sufficiently large. However, the radiance near the source contains more information about the scattering characteristic of the medium than the radiance far away from the source [2].

2. General Equations and Relations

In this section we include some general equations and indicate a useful relation connecting the angular radiance due to an isotropic point source with fluence caused by an unidirectional point source.

The stationary transport equation (RTE) for the radiance $G(\mathbf{r}, \hat{\mathbf{s}})$ caused by an isotropic point light source is given by

$$\hat{\mathbf{s}} \cdot \nabla G(\mathbf{r}, \hat{\mathbf{s}}) + \mu_t G(\mathbf{r}, \hat{\mathbf{s}}) = \mu_s \int_{S^2} f(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}') G(\mathbf{r}, \hat{\mathbf{s}}') \mathrm{d}\hat{\mathbf{s}}' + \frac{\delta(\mathbf{r})}{4\pi}, \quad (1)$$

where $\nabla = (\partial_x, \partial_y, \partial_z)$ is the gradient operator, $\mu_t = \mu_a + \mu_s$ is the total attenuation coefficient, μ_a is the absorption coefficient, and μ_s denotes the scattering coefficient. The unit vector $\hat{\mathbf{s}}$ specifies the direction of the photon propagation, and $f(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}')$ is the scattering phase function, which is normalized to unity.

The expected solution of Eq. (1) is the function $G(r, \mu)$, depending on the radial distance $r = |\mathbf{r}|$ and the solid angle $\mu = \hat{\mathbf{s}} \cdot \hat{\mathbf{r}}$. The derivations outlined below are based on the fact that the radiance due to an isotropic point source can be extracted from fluence, which is defined as

$$\Phi(\mathbf{r}) = \int G(\mathbf{r}, \hat{\mathbf{s}}) \mathrm{d}\hat{\mathbf{s}} \quad (2)$$

and generated by a unidirectional beam of light. More precisely, the radiance measured at the radial distance r in direction μ is, apart from the factor 4π, exactly the same as the fluence $\Phi(r, \mu_r)$ for $\mu_r = \mu$ caused by the unidirectional light source $S(\mathbf{r}, \hat{\mathbf{s}}) = \delta(\mathbf{r})\delta(\hat{\mathbf{s}} - \hat{\mathbf{z}})$.

Therefore, the isotropic point source at the right-hand side of Eq. (1) is replaced by the unidirectional point source given above. The reason for this replacement can be seen below. At the same time, we separate the radiance into its ballistic and diffuse contributions to obtain

$$\hat{\mathbf{s}} \cdot \nabla \psi(\mathbf{r}, \hat{\mathbf{s}}) + \mu_t \psi(\mathbf{r}, \hat{\mathbf{s}}) = \mu_s \int_{S^2} f(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}') \psi(\mathbf{r}, \hat{\mathbf{s}}') \mathrm{d}\hat{\mathbf{s}}' + \mu_s \frac{e^{-\mu_t r}}{r^2} \delta(\hat{\mathbf{r}} - \hat{\mathbf{z}}) f(\hat{\mathbf{s}} \cdot \hat{\mathbf{z}}), \quad (3)$$

which describes the diffuse light field, and

$$\psi_b(\mathbf{r}, \hat{\mathbf{s}}) = \frac{e^{-\mu_t r}}{r^2} \delta(\hat{\mathbf{r}} - \hat{\mathbf{z}}) \delta(\hat{\mathbf{s}} - \hat{\mathbf{z}}) \quad (4)$$

as the ballistic or uncollided portion of the radiance. Due to this separation, the source term in Eq. (3) becomes an exponentially weighted line source whose direction of radiation depends on the applied scattering phase function. The solution to the original RTE [Eq. (1)] is the fluence of Eq. (3) plus the uncollided part, where both terms must be divided by the solid angle 4π. Once the fluence has been found, we obtain Green's function of Eq. (1) in the form

$$G(r, \mu) = \frac{e^{-\mu_t r}}{4\pi r^2} \frac{\delta(\mu - 1)}{2\pi} + \frac{\Phi(r, \mu)}{4\pi}. \quad (5)$$

The unknown fluence

$$\Phi(r, \mu_r) = \int_{S^2} \psi(\mathbf{r}, \hat{\mathbf{s}}) \mathrm{d}\hat{\mathbf{s}}, \quad (6)$$

where μ_r must be replaced by μ, is derived in the next section.

3. Green's Function for the Fluence

In order to extract the fluence from Eq. (3), we apply at the beginning a three-dimensional Fourier transform with respect to the spatial position $\mathbf{r} = (x, y, z)$, yielding the RTE

$$(\mu_t + \mathrm{i}\mathbf{k} \cdot \hat{\mathbf{s}}) \psi(\mathbf{k}, \hat{\mathbf{s}}) = \mu_s \int_{S^2} f(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}') \psi(\mathbf{k}, \hat{\mathbf{s}}') \mathrm{d}\hat{\mathbf{s}}' + \mu_s \frac{f(\hat{\mathbf{s}} \cdot \hat{\mathbf{z}})}{\mu_t + \mathrm{i}k\mu_k}, \quad (7)$$

where $\mu_k = \hat{\mathbf{k}} \cdot \hat{\mathbf{z}}$. Next, multiplying both sides of Eq. (7) by the Legendre polynomials $P_l(\hat{\mathbf{s}} \cdot \hat{\mathbf{k}})$, integrating over all directions and defining moments $\psi_l(\mathbf{k})$ for $l = 0, 1, \ldots, N$ according to

$$\psi_l(\mathbf{k}) = \int \psi(\mathbf{k}, \hat{\mathbf{s}}) P_l(\hat{\mathbf{s}} \cdot \hat{\mathbf{k}}) \mathrm{d}\hat{\mathbf{s}}, \quad (8)$$

where N is assumed to be an odd number, leads to the following system of $N + 1$ linear equations:

$$\frac{l}{2l + 1} \mathrm{i}k\psi_{l-1}(\mathbf{k}) + \sigma_l \psi_l(\mathbf{k}) + \frac{l + 1}{2l + 1} \mathrm{i}k\psi_{l+1}(\mathbf{k}) = \mu_s f_l \frac{P_l(\mu_k)}{\mu_t + \mathrm{i}k\mu_k}, \quad (9)$$

where $\psi_{-1}(\mathbf{k}) = \psi_{N+1}(\mathbf{k}) = 0$ and $\sigma_l = \mu_a + (1 - f_l)\mu_s$. Moreover,

$$f_l = 2\pi \int_{-1}^{1} f(\mu) P_l(\mu) \mathrm{d}\mu \quad (10)$$

are the expansion coefficients of the rotationally invariant scattering phase function. In addition, the three-term recurrence relation

$$(\hat{\mathbf{s}}\cdot\hat{\mathbf{k}})P_l(\hat{\mathbf{s}}\cdot\hat{\mathbf{k}}) = \frac{l+1}{2l+1}P_{l+1}(\hat{\mathbf{s}}\cdot\hat{\mathbf{k}}) + \frac{l}{2l+1}P_{l-1}(\hat{\mathbf{s}}\cdot\hat{\mathbf{k}}) \tag{11}$$

as well as the integral

$$\int f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')P_l(\hat{\mathbf{s}}\cdot\hat{\mathbf{k}})\mathrm{d}\hat{\mathbf{s}} = f_l P_l(\hat{\mathbf{s}}'\cdot\hat{\mathbf{k}}) \tag{12}$$

have been considered in deriving Eq. (9). Setting $l = 0$ in Eq. (8) gives the lowest-order moment $\psi_0(\mathbf{k})$ corresponding to the fluence $\Phi(k, \mu_k)$. Note that the moments $\psi_l(\mathbf{k})$ for $l \geq 1$ cannot be used to constitute the complete angular dependent solution of Eq. (7). This can also be seen by inverting Eq. (8) such as

$$\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\psi_l(\mathbf{k}) = \int \psi(\mathbf{k},\hat{\mathbf{s}})\delta(\hat{\mathbf{s}}-\hat{\mathbf{k}})\mathrm{d}\hat{\mathbf{s}} = \psi(\mathbf{k},\hat{\mathbf{k}}), \tag{13}$$

which is the radiance as a function of the radial distance $k = |\mathbf{k}|$, whereas the angle of detection is fixed by $\hat{\mathbf{s}} = \hat{\mathbf{k}}$. The zero-order moment can be formally obtained from Eq. (9) via Cramer's rule and written in the form

$$\psi_0(\mathbf{k}) = \frac{\mu_s}{\mu_t + \mathrm{i}k\mu_k}\sum_{l=0}^{N}(2l+1)(-\mathrm{i}k)^l P_l(\mu_k) f_l \frac{D_l(k)}{D(k)}, \tag{14}$$

where $D_l(k)$ and $D(k)$ are even-degree polynomials with real coefficients and $\deg(D_l) < \deg(D)$. For example, if $N = 3$, we get the polynomial

$$D(k) = \begin{vmatrix} \sigma_0 & \mathrm{i}k & 0 & 0 \\ \mathrm{i}k & 3\sigma_1 & 2\mathrm{i}k & 0 \\ 0 & 2\mathrm{i}k & 5\sigma_2 & 3\mathrm{i}k \\ 0 & 0 & 3\mathrm{i}k & 7\sigma_3 \end{vmatrix} \tag{15}$$

$$\Phi(\mathbf{k}) = \frac{\mu_s}{\mu_t + \mathrm{i}k\mu_k}\frac{k\ \arctan(k/\mu_t) - 3g(\mu_s + \mathrm{i}k\mu_k)[1 - \mu_t\ \arctan(k/\mu_t)/k]}{k^2 - \mu_s k\ \arctan(k/\mu_t) - 3g\mu_a\mu_s[1 - \mu_t\ \arctan(k/\mu_t)/k]}. \tag{20}$$

as the determinant of a matrix that occurs when writing the system in Eq. (9) in matrix notation. For finding the polynomials $D_l(k)$, we at first have to replace the first column of the determinant by the corresponding zero vector. After that, we replace the lth column element in this vector by $1/(-\mathrm{i}k)^l$. Again, for $N = 3$, one obtains as an example the polynomial $D_1(k)$ as the determinant:

$$D_1(k) = \begin{vmatrix} 0 & \mathrm{i}k & 0 & 0 \\ 1/(-\mathrm{i}k)^1 & 3\sigma_1 & 2\mathrm{i}k & 0 \\ 0 & 2\mathrm{i}k & 5\sigma_2 & 3\mathrm{i}k \\ 0 & 0 & 3\mathrm{i}k & 7\sigma_3 \end{vmatrix}. \tag{16}$$

The polynomials given above as well as the product $k^l P_l(\mu_k)$ are even-degree polynomials with respect to the variable k, so that a partial fraction form can be obtained.

Thus, the fluence [Eq. (14)] in terms of $(N+1)/2$ partial fractions is found to be

$$\Phi(k, \mu_k) = \frac{\mu_s}{\mu_t + \mathrm{i}k_3}\sum_{\lambda_i>0}\sum_{l=0}^{N}(-\lambda_i)^l P_l(\mathrm{i}k_3/\lambda_i)\frac{A_l^{(i)}}{k^2 + \lambda_i^2}, \tag{17}$$

where $k_3 = k\mu_k$ and the constant $A_l^{(i)}$ follows from the decomposition

$$(2l+1)f_l\frac{D_l(k)}{D(k)} = \sum_{\lambda_i>0}\frac{A_l^{(i)}}{k^2 + \lambda_i^2}. \tag{18}$$

At this stage, the solution in Fourier space is completed and ready for inversion according to the Fourier integral

$$\Phi(\mathbf{r}) = \frac{1}{(2\pi)^3}\int \Phi(\mathbf{k})\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\mathrm{d}^3\mathbf{k}. \tag{19}$$

This calculation step is carried out in the next section.

We finish this section by including an exact result for the fluence [Eq. (17)] in the transformed space, which is usable for the verification. In the case of linear scattering, we have the phase function $f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}') = [1 + 3b(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')]/(4\pi)$. Here the limit $N \to \infty$ can be taken exactly, yielding the fluence [Eq. (17)] in closed-form representation:

4. Inverse Fourier Transformation

In this section, we consider at first the one-dimensional Fourier integral

$$f(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{P_l(\mathrm{i}k/\lambda)}{\mu_t + \mathrm{i}k}e^{\mathrm{i}kz}\mathrm{d}k, \tag{21}$$

because the obtained result can be applied for inverting the fluence [Eq. (17)] under the use of the convolution theorem. The integral [Eq. (21)] must be evaluated in terms of a polynomial differential operator. Based on the known transform pair,

$$\int_{-\infty}^{\infty} e^{-\mu_t z}\Theta(z)e^{-\mathrm{i}kz}\mathrm{d}z = \frac{1}{\mu_t + \mathrm{i}k}, \tag{22}$$

with $\Theta(z)$ being the unit step function, we evaluated the Fourier integral [Eq. (21)] in the form

$$f(z) = \mathcal{P}_l(D/\lambda)\{e^{-\mu_t z}\Theta(z)\}, \tag{23}$$

where $\mathcal{P}_l(D/\lambda)$ with $D \equiv \partial/\partial_z$ is a Legendre polynomial differential operator that is defined as

$$\mathcal{P}_l(D/\lambda) = 2^l \sum_{k=0}^{l} \binom{l}{k} \binom{\frac{l+k-1}{2}}{l} \frac{1}{\lambda^k} \frac{\partial^k}{\partial z^k}. \tag{24}$$

For the evaluation of Eq. (23), we need according to Eq. (24) the kth derivative of the function $g(z) = e^{-\mu_t z}\Theta(z)$. Using Leibnitz's rule or evaluating a few low-order derivatives explicitly, such as

$$g'(z) = \delta(z) - \mu_t g(z), \tag{25}$$

$$g''(z) = \delta^{(1)}(z) - \mu_t g'(z), \tag{26}$$

$$\vdots \tag{27}$$

$$g^{(k)}(z) = \delta^{(k-1)}(z) - \mu_t g^{(k-1)}(z), \tag{28}$$

yields the following closed-form expression:

$$g^{(k)}(z) = (-\mu_t)^k e^{-\mu_t z}\Theta(z) + \sum_{n=0}^{k-1} (-\mu_t)^{k-n-1}\delta^{(n)}(z), \tag{29}$$

where $\delta^{(n)}(z)$ is the nth derivative of the Dirac delta function $\delta(z)$. Below, we need the following convolution theorem involving the derivative of the Dirac delta function:

$$(D^n\delta * h)(z) = \int_{-\infty}^{\infty} h(z-u)\delta^{(n)}(u)\mathrm{d}u = h^{(n)}(z). \tag{30}$$

Based on the relations above, we obtain the inverse Fourier transform of Eq. (21) in the form

$$f(z) = (-1)^l P_l(\mu_t/\lambda)e^{-\mu_t z}\Theta(z) + \sum_{k=0}^{l} \frac{c_{kl}}{\lambda^k} \sum_{n=0}^{k-1} (-\mu_t)^{k-n-1}\delta^{(n)}(z), \tag{31}$$

with the Legendre polynomial coefficients

$$c_{kl} = 2^l \binom{l}{k} \binom{\frac{l+k-1}{2}}{l}. \tag{32}$$

Next, we consider the three-dimensional Fourier integral [Eq. (19)] to invert the transformed fluence [Eq. (17)]. This task can be accomplished by applying the convolution theorem in connection with the Fourier integrals

$$\int \frac{e^{-\lambda r}}{4\pi r} e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\mathrm{d}^3\mathbf{r} = \frac{1}{k^2 + \lambda^2},$$
$$\int \delta(x)\delta(y)f(z)e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\mathrm{d}^3\mathbf{r} = \frac{P_l(\mathrm{i}k_3/\lambda)}{\mu_t + \mathrm{i}k_3}, \tag{33}$$

where $f(z)$ is given in Eq. (31). Combining all relations above and performing some algebraic simplifications, we obtain for the inverse Fourier transform of Eq. (17) the function

$$\Phi(r,\mu_r) = \frac{\mu_s}{4\pi} \sum_{\lambda_i > 0} \sum_{l=0}^{N} A_l^{(i)} \lambda_i^l \Bigg[P_l(\mu_t/\lambda_i) \int_0^{\infty} \frac{e^{-\lambda_i \sqrt{r^2+\xi^2-2r\xi\mu_r}}}{\sqrt{r^2+\xi^2-2r\xi\mu_r}} e^{-\mu_t\xi}\mathrm{d}\xi + (-1)^l \sum_{k=0}^{l} \frac{c_{kl}}{\lambda_i^k} \sum_{n=0}^{k-1} (-\mu_t)^{k-n-1}\mathcal{D}_n(r,\mu_r) \Bigg], \tag{34}$$

corresponding to the fluence of Eq. (3). The above expression involves the computation of the nth order derivative of the function

$$\mathcal{D}_n(\mathbf{r}) = \frac{\partial^n}{\partial z^n} \frac{e^{-\lambda_i\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}}. \tag{35}$$

It can be shown either via induction or by Fourier analysis that the following closed form expression holds for an arbitrary $n \in \mathbb{N}_0$:

$$\mathcal{D}_n(\mathbf{r}) = (-1)^n n! \lambda_i^{n+1} \sum_{\nu} \frac{2^\nu(2\nu+1)(n/2+\nu/2)!}{(n/2-\nu/2)!(n+\nu+1)!} \times P_\nu(\mu_r)k_\nu(\lambda_i r), \tag{36}$$

where $r = \sqrt{\rho^2+z^2}$ and $\mu_r = z/r$. Furthermore, $k_\nu(x)$ is the modified spherical Bessel function of the second kind with $k_0(x) = x^{-1}e^{-x}$ and $k_1(x) = (x^{-1}+x^{-2})e^{-x}$. The summation index ν depends on the order n. That means that if n is an even number, the summation must be carried out for $\nu = 0, 2, \ldots, n$, resulting in $n/2+1$ terms. If n is an odd number, we have to summarize $(n+1)/2$ terms arising for the values $\nu = 1, 3, \ldots, n$. Upon the evaluation of the nth-order derivative, we have to introduce the variables $\rho = r\sqrt{1-\mu_r^2}$ and $z = r\mu_r$. The derived expression for the fluence [Eq. (34)] can now be inserted into Eq. (5) with the replacement $\mu_r = \mu$ to constitute the radiance of the original RTE [Eq. (1)].

It should be noted that it is, in principle, possible to avoid the work with the polynomial differential operator and the Dirac functions by inverting Eq. (14) directly under the use of the Fourier integral

$$\int k_l(\lambda r)P_l(\mu_r)e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\mathrm{d}^3\mathbf{r} = \frac{4\pi}{\lambda^{l+1}}\frac{(-\mathrm{i}k)^l P_l(\mu_k)}{k^2+\lambda^2}. \quad (37)$$

However, one decisive disadvantage is that the numerical evaluation of the convolution with the function $g(\mathbf{r}) = \delta(x)\delta(y)e^{-\mu_t z}\Theta(z)$ becomes a challenging and time-consuming task compared to the derived Green's function.

5. Fluence for the Isotropic Point Source

In this section, we also provide the fluence in terms of the ballistic and diffuse contributions for the case of an isotropic point light source. To this end, we remember that Eq. (14) divided by the factor 4π is the corresponding radiance. The spherically symmetric fluence $\Phi(k)$ is obtained via integration of the angular radiance in Fourier space. Equation (14) can be integrated by making use of the relation

$$\frac{1}{2}\int_{-1}^{1}\frac{P_l(x)}{\mu_t+\mathrm{i}kx}\mathrm{d}x = \frac{\mathrm{i}}{k}Q_l(\mathrm{i}\mu_t/k), \quad (38)$$

where $Q_l(z)$ is the Legendre function of the second kind [22]. Therefore, the fluence is found to be

$$\Phi(k) = \mu_s\frac{\mathrm{i}}{k}\sum_{l=0}^{N}(2l+1)(-\mathrm{i}k)^l Q_l(\mathrm{i}\mu_t/k)f_l\frac{D_l(k)}{D(k)}. \quad (39)$$

Here the inverse Fourier transform can be replaced by the one-dimensional spherical Hankel transform

$$W_{l-1}(z) = \sum_{n=1}^{l}\frac{1}{n}P_{n-1}(z)P_{l-n}(z). \quad (42)$$

Similar to the case of the angular radiance, the above integral can be carried out using a partial fraction decomposition as well as the Hankel transform pair:

$$\frac{1}{2\pi^2 r}\int_0^\infty \frac{\arctan(k/\mu_t)}{k^2+\lambda^2}\sin(kr)\mathrm{d}k = \frac{e^{-\lambda r}}{8\pi\lambda r}\ln\left|\frac{\mu_t+\lambda}{\mu_t-\lambda}\right| + \frac{e^{\lambda r}E_1[(\mu_t+\lambda)r] - e^{-\lambda r}E_1[(\mu_t-\lambda)r]}{8\pi\lambda r}, \quad (43)$$

where $E_1(x)$ is the exponential integral function [22]. The resulting fluence due to the isotropic light source in terms of the ballistic and diffuse contributions is found to be

$$\Phi(r) = \frac{e^{-\mu_t r}}{4\pi r^2} + \frac{\mu_s}{4\pi}\sum_{\lambda_i>0}\sum_{l=0}^{N}A_l^{(i)}\lambda_i^{l-1}\left[\frac{e^{-\lambda_i r}}{r}\mathcal{Q}_l(\mu_t/\lambda_i) + \frac{e^{\lambda_i r}E_1[(\mu_t+\lambda_i)r] - e^{-\lambda_i r}E_1[(\mu_t-\lambda_i)r]}{2r}P_l(\mu_t/\lambda_i)\right], \quad (44)$$

where $\mathcal{Q}_l(\mu_t/\lambda_i)$ is a Legendre function of the second kind, defined as

$$\mathcal{Q}_l(\mu_t/\lambda_i) = \frac{1}{2}P_l(\mu_t/\lambda_i)\ln\left|\frac{\mu_t+\lambda_i}{\mu_t-\lambda_i}\right| - W_{l-1}(\mu_t/\lambda_i). \quad (45)$$

Similar to what is shown above, we can give an exact solution for the case of linear scattering. In the limit $N\to\infty$, the fluence expression [Eq. (39)] converges to the function

$$\Phi(k) = \mu_s\frac{\arctan^2(k/\mu_t) - 3g[1-\mu_a\arctan(k/\mu_t)/k][1-\mu_t\arctan(k/\mu_t)/k]}{k^2-\mu_s k\arctan(k/\mu_t) - 3g\mu_a\mu_s[1-\mu_t\arctan(k/\mu_t)/k]}, \quad (46)$$

$$\Phi(r) = \frac{1}{2\pi^2 r}\int_0^\infty \Phi(k)\sin(kr)k\mathrm{d}k, \quad (40)$$

due to the given symmetry of the fluence. For evaluation of this integral, we make use of the identity

$$\frac{\mathrm{i}}{k}Q_l(\mathrm{i}\mu_t/k) = \frac{\arctan(k/\mu_t)}{k}P_l(\mathrm{i}\mu_t/k) - \frac{\mathrm{i}}{k}W_{l-1}(\mathrm{i}\mu_t/k), \quad (41)$$

where the second term is a polynomial of the form [22]

which can also be obtained via integration of Eq. (20) regarding the angular variables.

6. Numerical Experiments

In this section, we perform a few numerical experiments for verification of the derived solutions to the radiance and the fluence. In the figures shown below, we consider the commonly applied Henyey–Greenstein scattering phase function

$$f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}') = \frac{1}{4\pi}\frac{1-g^2}{[1+g^2-2g(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')]^{3/2}}, \quad (47)$$

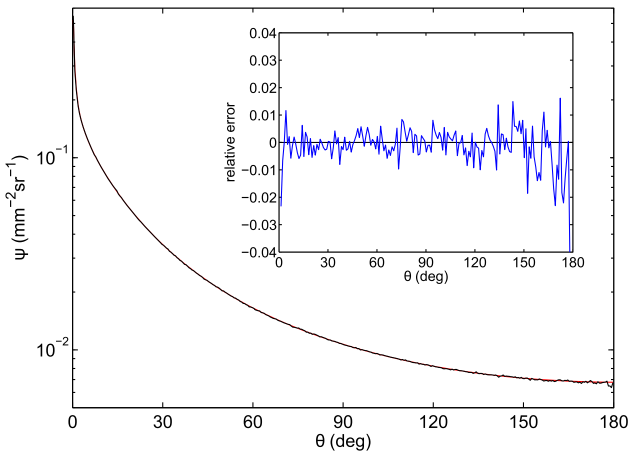

Fig. 1. Angle-resolved radiance in an infinitely extended anisotropically scattering medium. The optical properties are assumed to be $\mu_a = 0.01\ \text{mm}^{-1}$, $\mu_s' = 1\ \text{mm}^{-1}$, and $g = 0.8$. The radial distance to the isotropic point source is $r = 1.05$ mm.

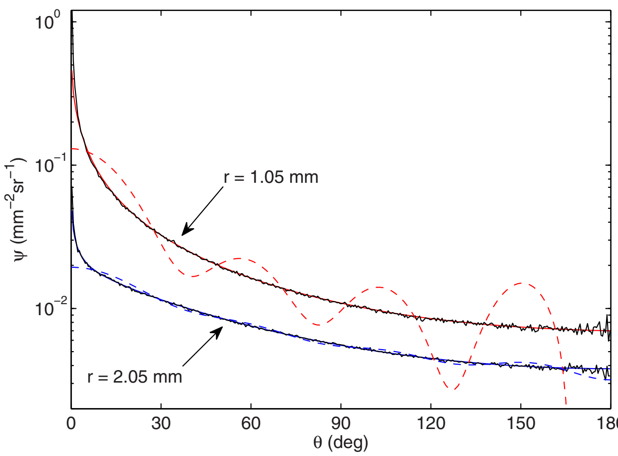

Fig. 3. Angle-resolved radiance in an infinitely extended anisotropically scattering medium. The optical properties are assumed to be $\mu_a = 0.01\ \text{mm}^{-1}$, $\mu_s' = 1\ \text{mm}^{-1}$, and $g = 0.6$.

where the corresponding expansion coefficients to be considered in the derived expressions are given by $f_l = g^l$. We note that other rotationally symmetric functions can be applied as well by making use of the relation in Eq. (10).

In Fig. 1, we evaluated the angle-resolved radiance due to an isotropic point light source located in an infinitely extended anisotropically scattering medium. The solid line corresponds to the derived Eq. (34), whereas the noisy curve is the result obtained from the MC simulation. In addition, we computed the relative differences between both methods and included them in the inset of Fig. 1.

In Fig. 2, we compared the angle-resolved radiance due to the point light source at three different radial distances. Again, the solid lines correspond to Eq. (5), whereas the noisy curves are obtained from the MC simulation. In Fig. 3, we undertake a comparison with respect to the convergence and the numerical stability. To this end, we consider the angle-resolved radiance for the order $N = 7$ at two radial distances to the point source. We have also included the results of the only available analytical solutions (dashed lines) to this problem, which have been derived recently [20,21]. We evaluated this solution for the same order, $N = 7$. Figures 1–3 show that the derived radiance expression [Eq. (34)] is in good agreement with the data generated by the MC simulation. In particular, Fig. 3 indicates the resulting improvements with respect to the convergence and the numerical stability compared to the existing models. We note that the shown oscillations cannot be removed by increasing the approximation order N. In fact, the solution becomes more and more unstable.

Finally, we evaluated the fluence as a function of the radial distance according to Eq. (44) for the case of an anisotropically scattering medium, which is shown in Fig. 4. The solid line corresponds to

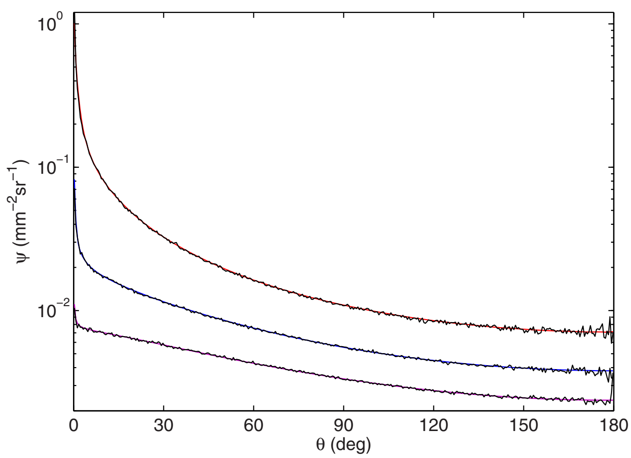

Fig. 2. Angle-resolved radiance in an infinitely extended anisotropically scattering medium. The optical properties are assumed to be $\mu_a = 0.01\ \text{mm}^{-1}$, $\mu_s' = 1\ \text{mm}^{-1}$, and $g = 0.6$. From top to bottom, the curves correspond with $r = 1.05$ mm, $r = 2.05$ mm, and $r = 3.05$ mm.

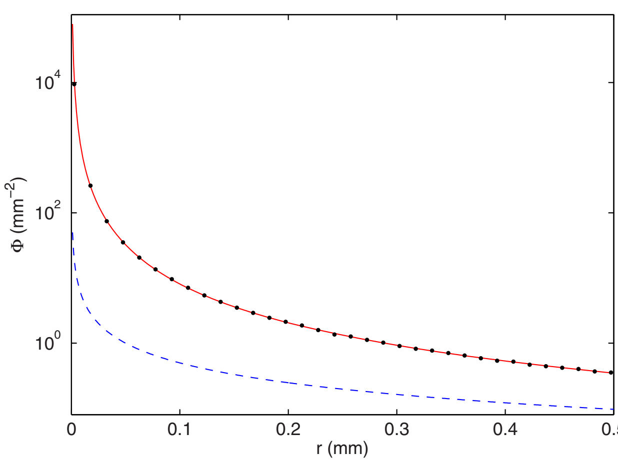

Fig. 4. Fluence as a function of the radial distance to the isotropic point light source in an infinitely extended anisotropically scattering medium. The optical properties are assumed to be $\mu_a = 0.01\ \text{mm}^{-1}$, $\mu_s' = 0.2\ \text{mm}^{-1}$, and $g = 0.8$.

Eq. (44), whereas the filled dots are the fluence data generated by the MC simulation. In addition, we included the fluence obtained from the DA, corresponding to the dashed line. The derived fluence expression [Eq. (44)] is in good agreement with the fluence generated by the MC method. In contrast, the fluence in the DA shows large differences from the exact transport theory solutions due to the relatively small distances to the isotropic point light source. We note that the differences between the transport theory solutions and the DA are significantly larger in the case of the angular radiance than in the case of the fluence. We therefore omitted the radiance in the DA within the first three figures.

7. Discussion

In conclusion, we derived an analytical solution for the radiance in an infinitely extended anisotropically scattering medium that is illuminated by an isotropic light point source. The obtained expressions can, for example, be directly applied to radiance measurements in biological media to determine the optical properties, which has been shown in [13,14], as well as to verification of other solution approaches. We showed the resulting improvements with respect to the convergence and the numerical stability compared to recently derived RTE solutions dealing with the same problem. The computation time needed for evaluation of the angle-resolved radiance for 360 angles using a nonoptimized single-core code is in the range of several milliseconds.

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References

1. M. S. Patterson, B. Chance, and C. Wilson, "Time resolved reflectance and transmittance for the non-invasive measurement of tissue optical properties," Appl. Opt. **28**, 2331–2336 (1989).
2. K. Rinzema, B. J. Hoenders, H. A. Ferwerda, and J. J. Ten Bosch, "Analytic calculation of the radiance in an anisotropically scattering turbid medium close to a source," Pure Appl. Opt. **4**, 629–642 (1995).
3. F. Martelli, M. Bassani, L. Alianelli, L. Zangheri, and G. Zaccanti, "Accuracy of the diffusion equation to describe photon migration through an infinite medium: numerical and experimental investigation," Phys. Med. Biol. **45**, 1359–1373 (2000).
4. H. Gao, L. Phan, and Y. Lin, "Parallel multigrid solver of radiative transfer equation for photon transport via graphics processing unit," J. Biomed. Opt. **17**, 0960041 (2012).
5. H. Erkol, A. Demirkiran, N. Uluc, and M. B. Unlu, "Analytical reconstruction of the bioluminescent source with priors," Opt. Express **22**, 19758–19773 (2014).
6. N. J. McCormick and A. Kaskas, "Isotropic spherical source analysis for ocean optics," Appl. Opt. **39**, 4902–4910 (2000).
7. R. Sanchez and N. J. McCormick, "Analytic beam spread function for ocean optics applications," Appl. Opt. **41**, 6276–6288 (2002).
8. S. Chandrasekhar, *Radiative Transfer* (Dover, 1960).
9. M. F. Modest, *Radiative Heat Transfer* (Academic, 2003).
10. M. M. R. Williams, "A synthetic scattering kernel for particle transport in absorbing media with anisotropic scattering," J. Phys. D **11**, 2455–2463 (1978).
11. F. Khalafi and M. M. R. Williams, "Energy deposition from an anisotropic point source in an infinite medium," Radiat. Eff. **46**, 175–180 (1980).
12. B. D. Ganapol, "A consistent theory of neutral particle transport in an infinite medium," Transp. Theory Stat. Phys. **29**, 43–68 (2000).
13. L. C. L. Chin, W. M. Whelan, and I. A. Vitkin, "Information content of point radiance measurements in turbid media: implications for interstitial optical property quantification," Appl. Opt. **45**, 2101–2114 (2006).
14. S. Grabtchak and W. M. Whelan, "Separation of absorption and scattering properties of turbid media using relative spectrally resolved cw radiance measurements," Biomed. Opt. Express **3**, 2371–2380 (2012).
15. G. W. Faris, "P_N approximation for frequency-domain measurements in scattering media," Appl. Opt. **44**, 2058–2071 (2005).
16. C. Baltes and G. W. Faris, "Frequency domain measurements on turbid media with strong absorption using the P_N approximation," Appl. Opt. **48**, 2991–3000 (2009).
17. E. d'Eon, "Rigorous asymptotic and moment-preserving diffusion approximations for generalized linear Boltzmann transport in arbitrary dimension," Transp. Theory Stat. Phys. **42**, 237–297 (2014).
18. M. L. Shendeleva, "Single-scattering solutions to radiative transfer in infinite turbid media," J. Opt. Soc. Am. A **30**, 2169–2174 (2013).
19. M. Machida, "Singular eigenfunctions for the three-dimensional radiative transport equation," J. Opt. Soc. Am. A **31**, 67–74 (2014).
20. V. A. Markel, "Modified spherical harmonics method for solving the radiative transport equation," Waves Random Complex Media **14**, L13–L19 (2004).
21. A. Liemert and A. Kienle, "Analytical Green's function of the radiative transfer radiance for the infinite medium," Phys. Rev. E **83**, 036605 (2011).
22. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, 2007).